Master OAM – Proposition de projet

à l'Institut de Physique de Nice
CNRS & Université Côte d'Azur

How do you connect a divergent series to a physical observable?

Theoretical/Numerical study

Introduction.— The solution to physical problems is often obtained through perturbation techniques, presenting the solution as a series in a small parameter. In quantum mechanics, a classic example involves determining the lowest energy level, or ground state, of a system by solving the Schrödinger equation. Analytical solutions are available for only a few potentials, such as the harmonic oscillator or the hydrogen atom. For other potentials lacking an analytical solution, numerical approaches are commonly employed.

However, one approach is to reframe the problem by expressing the potential as the sum of a term with a known analytical solution and a term treated as a small perturbation, governed by a small parameter, denoted as ϵ. The goal is to reconstruct the analytic solution for the ground-state energy, $E(\epsilon)$, as a series in ϵ. Despite successful instances, it is often discovered that the series has a zero radius of convergence, and the sum of all terms appears infinite.

What one has truly achieved is an asymptotic series for the real function $E(\epsilon)$. This represents the function and allows the recovery of the "real number".

Making sense of asymptotic series remains an important program in both mathematics and physics. In physics, many modern theories rely on perturbative expansions, and understanding the nature of a divergent series is crucial for making sense of the theories themselves.

Familiarity with divergent series provides an introduction to the broader subject known as resurgence theory. In many problems, there exist a multitude of simple systems around which perturbative solutions are developed. For a long time, it was believed that these perturbative series were isolated and unrelated. Recently, resurgence theory has suggested that this may not be the case. It has shown that the perturbative expansion around one of these simple systems can provide valuable information about other possible perturbative expansions. This opens the possibility of finding a sensible way to define theories themselves.

Objectives.— During this project, the student will have the opportunity to explore and comprehend the nature of the usual perturbative expansion commonly employed in quantum mechanics, recognizing it as a divergent series. The focus will be on understanding what constitutes an asymptotic series and how one can interpret it, employing techniques such as Padé resummation and Borel resummation.

The journey will commence with a straightforward example—an integral that can be viewed as a zero-dimensional quantum field theory [1]. From there, the exploration will progress towards more intricate examples, such as the quartic oscillator [2]. Depending on the student's interests and the available time, there is also the option to delve into more complex and intriguing problems, such as the double well scenario.

Contact.— Mario Gattobigio: `mario.gattobigio@univ-cotedazur.fr`, +33 (0)4 89 15 27 76

[1] Marcos Mariño, *An introduction to resurgence in quantum theory*, lecture notes.
[2] C. M. Bender and T. T. Wu, Anharmonic oscillator, Phys. Rev. 184 (1969) 1231–1260.